Gaussian Elimination

CS6015 : Linear Algebra

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The Gaussian Elimination Method

- The Gaussian elimination method is a technique for solving systems of linear equations of any size.
- The operations of the Gaussian elimination method are:
    1. Interchange any two equations.
    2. Replace an equation by a nonzero constant multiple of itself.
    3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Multiply the equation by 1/2

Example

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<u>Solution</u>

- First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Multiply the equation by 1/2

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Replace by the sum of – 3 x (the first equn.) + (second equation)

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
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<u>Solution</u>

- Next, we eliminate the variable x from all equations except the first:

$$x + 2y + 3z = 11$$
$$2y - 4z = -6$$
$$-x + y + 2z = 2$$

Replace by the sum of – 3 x (the first equn.) + (second equation)

Example

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$$2x + 4y + 6z = 22$$
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$$x + 2y + 3z = 11$$
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Replace by the sum of the (first equation) + (third equation)

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$$2x + 4y + 6z = 22$$
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<u>Solution</u>

- Then we transform so that the coefficient of y in the second equation is 1:

$$x + 2y + 3z = 11$$
$$2y - 4z = -6$$
$$3y + 5z = 13$$

Multiply the second equation by 1/2

Example

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<u>Solution</u>

- We now eliminate y from all equations except the second:

$$x + 2y + 3z = 11$$
$$y - 2z = -3$$
$$3y + 5z = 13$$

← **Replace by the sum of the first equation + (–2) x (the second equation)**

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate y from all equations except the second:

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Replace by the sum of the first equation + (–2) x (the second equation)

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate y from all equations except the second:

$$\begin{aligned} x \quad\quad + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$$

Replace by the sum of the third equation + (–3) x (the second equation)

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate y from all equations except the second:

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Replace by the sum of the third equation + (–3) x (the second equation)

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- Now we transform so that the coefficient of z in the third equation is 1:

Triangular system

$$\begin{aligned} x \qquad + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$$

Multiply the third equation by 1/11

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- Now we transform so that the coefficient of z in the third equation is 1:

$$\begin{aligned} x \quad\quad + 7z &= 17 \\ y - 2z &= -3 \\ \boxed{z = 2} \end{aligned}$$

Multiply the third equation by 1/11

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

Back substitution $x \quad + 7z = 17$ ← **Putting value of z in the first equation**

$$y - 2z = -3$$

$$z = 2$$

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Solution

- We now eliminate z from all equations except the third:

Back substitution

$$x \quad = 3$$
$$y - 2z = -3$$
$$z = 2$$

Putting value of z in the first equation

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

<u>Solution</u>

- We now eliminate z from all equations except the third:

Back substitution

$$x \qquad\qquad = 3$$
$$y - 2z = -3$$
$$z = 2$$

Putting value of z in the second equation

Example

- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
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<u>Solution</u>

- We now eliminate z from all equations except the third:

Back substitution

$$x = 3$$
$$y = 1$$
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Putting value of z in the second equation

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- Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
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<u>Solution</u>

- Thus, the solution to the system is $x = 3$, $y = 1$, and $z = 2$.

$$x \qquad = 3$$
$$y \qquad = 1$$
$$z = 2$$

Row-Reduced Form of a Matrix

- Each row consisting entirely of zeros lies below all rows having nonzero entries.
- The first nonzero entry in each nonzero row is 1 (called a leading 1).
- In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
- If a column contains a leading 1, then the other entries in that column are zeros.

Row Operations

1. Interchange any two rows.

2. Replace any row by a nonzero constant multiple of itself.

3. Replace any row by the sum of that row and a constant multiple of any other row.

Terminology for the Gaussian Elimination Method

Unit Column

- A column in a coefficient matrix is in unit form if one of the entries in the column is a 1 and the other entries are zeros.

Pivoting

- The sequence of row operations that transforms a given column in an augmented matrix into a unit column.

Notation for Row Operations

- Letting R_i denote the i-th row of a matrix, we write

<u>Operation 1</u>: $R_i \leftrightarrow R_j$ to mean:
Interchange row i with row j.

<u>Operation 2</u>: cR_i to mean:
replace row i with c times row i.

<u>Operation 3</u>: $R_i + aR_j$ to mean:
Replace row i with the sum of row i and a times row j.

Example

- Pivot the matrix about the circled element

$$\begin{bmatrix} \boxed{3} & 5 & 9 \\ 2 & 3 & 5 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{cc|c} 3 & 5 & 9 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & 5/3 & 3 \\ 2 & 3 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 5/3 & 3 \\ 2 & -1/3 & -1 \end{array}\right]$$

The Gaussian Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows, if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Augmented Matrices

- Matrices are rectangular arrays of numbers that can aid us by eliminating the need to write the variables at each step of the reduction.
- For example, the system

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

may be represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

Augmented matrix [C|B]

Coefficient Matrix [C]

Matrices and Gaussian Elimination

- Every step in the Gaussian elimination method can be expressed with matrices, rather than systems of equations, thus simplifying the whole process:
- Steps expressed as systems of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

- Steps expressed as augmented matrices:

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\boldsymbol{R_1' = \frac{1}{2} R_1}$$

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\boldsymbol{R_2' = R_2 - 3R_1}$$

$$x + 2y + 3z = 11$$
$$2y - 4z = -6$$
$$-x + y + 2z = 2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\boldsymbol{R_3' = R_3 + R_1}$$

$$x + 2y + 3z = 11$$
$$2y - 4z = -6$$
$$3y + 5z = 13$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\boldsymbol{R_2' = \frac{1}{2} R_2}$$

$$R_2' = \frac{1}{2}R_2$$

$$\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \qquad \left[\begin{array}{cccc} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$R_1' = R_1 - 2R_2$$

$$\begin{aligned} x \quad + 7z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned} \qquad \left[\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$R_3' = R_3 - 3R_2$$

$$\begin{aligned} x \quad + 7z &= 11 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned} \qquad \left[\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right]$$

$$R_3' = \frac{1}{11}R_3$$

$$\begin{aligned} x \quad + 7z &= 11 \\ y - 2z &= -3 \\ z &= 2 \end{aligned} \qquad \left[\begin{array}{cccc} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1' = R_1 - 7R_3$$

$$R_1' = R_1 - 7R_3$$

$$\begin{aligned} x \qquad\quad &= 3 \\ y - 2z &= -3 \\ z &= 2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2' = R_2 + 2R_3$$

$$\begin{aligned} x \qquad &= 3 \\ y \quad &= -3 \\ z &= 2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row Reduced Form of the Matrix

Thus, the solution to the system is $x = 3$, $y = 1$, and $z = 2$.

Gaussian Elimination in the case of unique solution

- With a full set of n pivots, there is only one solution.
- The system is non singular, and it is solved by forward elimination and back-substitution.

Under what circumstances could the process break down?

- But if a **zero appears in a pivot position**, elimination has to stop—either temporarily or permanently. The system might or might not be singular.
- If the first coefficient is zero, in the upper left corner, the elimination of u from the other equations will be impossible.
- The same is true at every intermediate stage. Notice that a zero can appear in a pivot position, even if the original coefficient in that place was not zero.
- Roughly speaking, **we do not know whether a zero will appear until we try**, by actually going through the elimination process.
- In many cases this problem can be cured, and elimination can proceed. Such a system still counts as nonsingular; it is only the algorithm that needs repair.
- In other cases a breakdown is unavoidable. Those incurable systems are **singular**, they have **no solution** or **else infinitely many**, and a full set of pivots cannot be found.

Systems of Linear Equations: Underdetermined and Overdetermined systems

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow[\boldsymbol{R_3 - 2R_1}]{\boldsymbol{R_2 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & -7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\boldsymbol{-\frac{1}{7}R_2}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[\boldsymbol{R_3 + R_2}]{\boldsymbol{R_1 - 2R_2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Observe that row three reads $0 = 0$, which is true but of no use to us.

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$\begin{aligned} x + 2y - 3z &= -2 \\ 3x - y - 2z &= 1 \\ 2x + 3y - 5z &= -3 \end{aligned}$$

<u>Solution</u>

- This last augmented matrix is in row-reduced form.
- Interpreting it as a system of equations gives a system of two equations in three variables x, y, and z:

$$\begin{aligned} x - z &= 0 \\ y - z &= -1 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

Solution

- Let's single out a single variable –say, z– and solve for x and y in terms of it.
- If we assign a particular value of z –say, $z = 0$ we obtain $x = 0$ and $y = -1$, giving the solution $(0, -1, 0)$.

$$x - z = 0 \qquad x = z = (0) = 0$$
$$y - z = -1 \qquad y = z - 1 = (0) - 1 = -1$$

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

<u>Solution</u>

- Let's single out a single variable –say, z– and solve for x and y in terms of it.
- If we instead assign $z = 1$, we obtain the solution $(1, 0, 1)$.

$$x - z = 0 \qquad\qquad x = z \qquad = (1) = 1$$
$$y - z = -1 \qquad\qquad y = z - 1 = (1) - 1 = 0$$

A System of Equations with an Infinite Number of Solutions

- Solve the system of equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

Solution

- Let's single out a single variable –say, z– and solve for x and y in terms of it.
- In general, we set $z = t$, where t represents any real number (called the parameter) to obtain the solution $(t, t - 1, t)$.

$$x - z = 0 \qquad\qquad x = z \quad = (t) = t$$
$$y - z = -1 \qquad\qquad y = z - 1 = (t) - 1 = t - 1$$

A System of Equations That Has No Solution

- Solve the system of equations given by

$$x + y + z = 1$$
$$3x - y - z = 4$$
$$x + 5y + 5z = -1$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \xrightarrow[\boldsymbol{R_3 - R_1}]{\boldsymbol{R_3 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$\boldsymbol{R_3 + R_2}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

A System of Equations That Has No Solution

- Solve the system of equations given by

$$x + y + z = 1$$
$$3x - y - z = 4$$
$$x + 5y + 5z = -1$$

<u>Solution</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

- Observe that row three reads

$0x + 0y + 0z = -1$ or $0 = -1$!

- We therefore conclude the system is inconsistent and has no solution.

Systems with no Solution

- If there is a row in the augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the system of equations has no solution.

Theorem

a. If the number of equations is greater (over-determined system) than or equal to the number of variables in a linear system, then one of the following is true:
   i. The system has no solution.
   ii. The system has exactly one solution.
   iii. The system has infinitely many solutions.

b. If there are fewer equations than variables (under-determined system) in a linear system, then the system either has no solution or it has infinitely many solutions.

Cost of elimination

How many separate arithmetical operations does elimination require, for n equations in n unknowns?

- Ignore the right-hand sides of the equations, and count only the operations on the left. These operations are of two kinds.
- We divide by the pivot to find out what multiple (say ℓ) of the pivot equation is to be subtracted.
- When we do this subtraction, we continually meet a "*multiply-subtract*" combination; the terms in the pivot equation are multiplied by ℓ, and then subtracted from another equation.
- **We call each division, and each multiplication-subtraction, one operation.**
- In column 1, ***it takes n operations for every zero we achieve***—one to find the multiple ℓ, and the other to find the new entries along the row.

Cost of elimination (contd.)

- There are $n-1$ rows underneath the first one, so the first stage of elimination needs $n(n-1)=n^2-n$ operations.

When the elimination is down to k equations, only k^2-k operations are needed to clear out the column below the pivot—by the same reasoning that applied to the first stage, when k equaled n. Altogether, the total number of operations is the sum of k^2-k over all values of k from 1 to n:

Left side $$(1^2+\cdots+n^2)-(1+\cdots+n)=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}$$
$$=\frac{n^3-n}{3}.$$

- If $\boldsymbol{n}$ is at all large, a good estimate for the number of operations is $\frac{1}{3}\boldsymbol{n}^3$.
- Back-substitution is considerably faster. The last unknown is found in only one operation (a division by the last pivot). The second to last unknown requires two operations, and so on. Then the total for back-substitution is $1+2+\cdots+n$.

Cost of elimination (contd.)

Forward elimination also acts on the right-hand side (subtracting the same multiples as on the left to maintain correct equations). This starts with $n-1$ subtractions of the first equation. Altogether ***the right-hand side is responsible for n^2 operations***—much less than the $n^3/3$ on the left. The total for forward and back is

$$\textbf{Right side} \qquad [(n-1)+(n-2)+\cdots+1]+[1+2+\cdots+n]=n^2.$$

Special Notes :

There now exists a method that requires only $Cn^{log_2 7}$ multiplications! It depends on a simple fact: Two combinations of two vectors in two-dimensional space would seem to take 8 multiplications, but they can be done in 7. That lowered the exponent from $\log_2 8$, which is 3, to $\log_2 7 \approx 2.8$. This discovery produced tremendous activity to find the smallest possible power of n. The exponent finally fell (at IBM) below 2.376. Fortunately for elimination, the constant C is so large and the coding is so awkward that the new method is largely (or entirely) of theoretical interest. The newest problem is the cost with many processors in parallel.